

Bang-bang property of time optimal controls for some semilinear heat equation

Lijuan Wang* and Qishu Yan†

Abstract

In this paper, we derive a bang-bang property of a kind of time optimal control problem for some semilinear heat equation on bounded C^2 domains (of the Euclidean space), with homogeneous Dirichlet boundary condition and controls distributed on an open and non-empty subset of the domain where the equation evolves.

2010 Mathematics Subject Classifications. 35K58, 49J20, 49J30, 93B07

Keywords. time optimal controls, bang-bang property, semilinear heat equation, observability estimate from measurable sets

1 Introduction

In this paper, we assume that Ω is a bounded domain in $\mathbb{R}^d, d \geq 1$, with a C^2 boundary $\partial\Omega$; ω is an open and non-empty subset of Ω ; χ_ω is the characteristic function of the set ω . Define, for an arbitrarily fixed $M > 0$, $q \in (d, \infty)$ if $d \geq 2$ and $q \in [2, \infty)$ if $d = 1$, a control constraint set:

$$\mathcal{U} \triangleq \{u : [0, +\infty) \rightarrow L^q(\Omega) \text{ is measurable} : \|u(t)\|_{L^q(\Omega)} \leq M \text{ for almost all } t > 0\}.$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function holding the following properties:

- (H_1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz;
- (H_2) $f(y)y \geq 0$ for all $y \in \mathbb{R}$.

We arbitrarily fix a $y_0 \in C_0(\Omega) \setminus \{0\}$. Here $C_0(\Omega) \triangleq \{y \in C(\overline{\Omega}) : y = 0 \text{ on } \partial\Omega\}$.

*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China; e-mail: ljwang.math@whu.edu.cn. This work was supported by the National Natural Science Foundation of China under grant 11371285.

†School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China. e-mail: yan-qishu@whu.edu.cn.

The controlled semilinear heat equation under consideration is as follows:

$$\begin{cases} y_t - \Delta y + f(y) = \chi_\omega u & \text{in } \Omega \times (0, +\infty), \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $u \in \mathcal{U}$. For each $u \in \mathcal{U}$ and $T > 0$, Equation (1.1) has a unique solution in $C([0, T]; C_0(\Omega))$, denoted it by $y(\cdot; y_0, u)$ (see Proposition 2.3). Throughout the paper, we will omit the variables x and t for functions of (x, t) and the variable x for functions of x , if there is no risk of causing any confusion.

Define the following admissible set of controls:

$$\mathcal{U}_{ad} \triangleq \{u \in \mathcal{U} : y(\cdot; y_0, u) \in C([0, T]; C_0(\Omega)) \text{ and } y(T; y_0, u) = 0 \text{ for some } T > 0\}.$$

Each control in \mathcal{U}_{ad} is called an admissible control. Because of the properties (H_1) and (H_2) , it is proved that the set \mathcal{U}_{ad} is not empty (see Proposition 3.3). For each $u \in \mathcal{U}_{ad}$, we set

$$T(u) \triangleq \min\{T : y(T; y_0, u) = 0\}.$$

The above minimum can be reached because the solution $y(\cdot; y_0, u)$ can be treated as a continuous function from $[0, +\infty)$ to $C_0(\Omega)$.

Now, the time optimal control problem under consideration is as follows:

$$(P) \quad T^* \triangleq \inf_{u \in \mathcal{U}_{ad}} T(u).$$

In this problem, the number T^* is called the optimal time; a control $u^* \in \mathcal{U}_{ad}$, with $y(T^*; y_0, u^*) = 0$, is called a time optimal control (or optimal control for simplicity). It is proved that (P) has optimal controls (see Proposition 3.4).

The main result of this paper is stated as follows:

Theorem 1.1. *The problem (P) holds the bang-bang property: any optimal control u^* satisfies that $\|u^*(t)\|_{L^q(\Omega)} = M$ for a.e. $t \in (0, T^*)$.*

The bang-bang property is one of the most important and interesting properties of time optimal control problems. To our best knowledge, this property was first built up, via a smart construction manner, for time optimal control problems of linear abstract equations in Banach spaces by H.O.Fattorini (see [3]). But in the context of the distributed control of the heat equation, the results in [3] only apply for the very special case where the control is distributed everywhere in the domain, i.e. $\omega = \Omega$. Since then, such property has been studied for time optimal control problems of linear and semilinear parabolic differential equations, where controls are distributed everywhere in the domain Ω , in many papers (see e.g. [1], [2], [4], [10], [19] and the references therein). It is worth mentioning the following studies on the bang-bang property for time optimal control problems of parabolic equations with control restricted on a subset ω of Ω . In [15], the bang-bang property was obtained for a time optimal control problem of the heat

equation with pointwise boundary control constraints. Some assumptions on the control bound was imposed there. Partially motivated by the study [15], the authors in [11] first realized that the bang-bang property can be implied by the null-controllability of the system with controls restricted over any set of positive measure in a time interval. They further proved such null-controllability for one-dimensional heat equation with boundary controls. The aforementioned null-controllability was established in [16] for internally controlled heat equations over domains $\Omega \subset \mathbb{R}^d$, $d \geq 1$. As a consequence, the bang-bang property holds for time optimal control problems of the corresponding equations. Motivated by these results, the authors in [14], [7] and [8] realized that the bang-bang property for time optimal controls of semilinear heat equations can be obtained by combining a strategy based on null controllability of the system, where the control functions act on a measurable set, and a fixed point argument. The main differences between this work and [14], [7], [8] are as follows: In [14], the semilinear term is globally Lipschitz and satisfies the sign condition (H_2) . Therefore, the bang-bang property is global, i.e., it holds for any initial datum and any control bound. In [7] and [8], the controlled equations are Burgers equation and a general semilinear heat equation without sign condition (H_2) . There the bang-bang property is local with respect to the initial datum and the control bound. In our paper, the semilinear term is locally Lipschitz with sign condition, and the global bang-bang property is derived. Hence, this paper is a nontrivial extension of the above-mentioned works. It should be pointed out that if $y_0 \in L^\infty(\Omega)$, by the similar arguments as those in this paper, the global bang-bang property still holds.

It should be pointed out that in [16], [14], [7], [8] and this paper, the controlled systems are time invariant and targets are a single point in state space. The proofs rely on the translation invariant of the systems. This method does not work for the time varying systems. In [17], when the controlled systems read as follows:

$$\begin{cases} y_t - \Delta y + (a_1(x) + a_2(t))y = \chi_\omega u & \text{in } \Omega \times (0, +\infty), \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

and the target was a single point, the bang-bang property of time optimal controls was obtained. The strategy used there was totally different from the existing works. Instead of studying directly the time optimal control problem, the authors started from studying properties of optimal norms for norm optimal control problems. However, when the potential $a_1(x) + a_2(t)$ is replaced by $a(x, t)$, whether the bang-bang property holds or not is still an open problem.

When the target set is a ball, there is another possible way to get bang-bang properties for time optimal control problems of differential equations. More precisely, one can expect to derive bang-bang properties from the Pontryagin maximum principle and unique continuation property for the corresponding equations. Here, we would like to mention some related papers [18], [5] and [6].

To prove Theorem 1.1, we need an observability estimate from a measurable set for heat equations with lower terms, and then use the Kakutani fixed point theorem. To this end, we

introduce the following heat equation:

$$\begin{cases} \varphi_t - \Delta\varphi + a\varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(0) \in L^2(\Omega). \end{cases} \quad (1.2)$$

Here, $a \in L^\infty(\Omega \times (0, T))$. Then

Lemma 1.2. *[14] Let $E \subset (0, T)$ be a subset of positive measure. Then any solution to Equation (1.2) satisfies the following observability estimate:*

$$\|\varphi(T)\|_{L^2(\Omega)} \leq e^{C(\Omega, \omega, E)} e^{C(\Omega, \omega)(1+T\|a\|_{L^\infty(\Omega \times (0, T))} + \|a\|_{L^\infty(\Omega \times (0, T))}^{2/3})} \int_{\omega \times E} |\varphi(x, t)| dx dt. \quad (1.3)$$

Here and throughout this paper, we shall use $C(\dots)$ to denote several positive constants depending on what are enclosed in the bracket.

Remark 1.3. *The estimate (1.3) is an observability inequality from a measurable set. It was first established for the case that Ω is convex in [13]. In [14], the convexity assumption on Ω was successfully dropped. Later on, the regularity assumption on the potential a was relaxed to $a \in L^\infty(0, T; L^q(\Omega))$ in [7].*

The rest of the paper is organized as follows: In Section 2, we provide some existence results and prior estimates of solutions for parabolic equations. These results are required to obtain the bang-bang property of the problem (P). In Section 3, we give the proof of Theorem 1.1.

2 Preliminary results

In this section, we shall present three results that will be essential for the proof of the bang-bang property. The first and the second results are concerned with the existence and energy estimates of solutions for linear heat equations.

Lemma 2.1. *Let $g \in L^q(0, T; L^q(\Omega))$ and $z_0 \in C_0(\Omega)$. Then the equation*

$$\begin{cases} z_t - \Delta z = g & \text{in } \Omega \times (0, T), \\ z = 0 & \text{on } \partial\Omega \times (0, T), \\ z(0) = z_0 & \text{in } \Omega \end{cases} \quad (2.1)$$

has a unique solution, denoted by z , in

$$L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega)) \cap C([0, T]; C_0(\Omega)).$$

Moreover,

$$\|z\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T)(\|g\|_{L^q(0, T; L^q(\Omega))} + \|z_0\|_{C(\overline{\Omega})}).$$

Proof. It is obvious that (2.1) has a unique solution, denoted by z , in $L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))$. Moreover,

$$z = z_1 + z_2, \quad (2.2)$$

where z_1 and z_2 are solutions to equations

$$\begin{cases} (z_1)_t - \Delta z_1 = g & \text{in } \Omega \times (0, T), \\ z_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ z_1(0) = 0 & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} (z_2)_t - \Delta z_2 = 0 & \text{in } \Omega \times (0, T), \\ z_2 = 0 & \text{on } \partial\Omega \times (0, T), \\ z_2(0) = z_0 & \text{in } \Omega, \end{cases}$$

respectively. On one hand, it follows from Sobolev embedding theorem and L^p -theory for parabolic equation (See e.g. Theorem 9.1 of Chapter 4 in [9]) that

$$\|z_1\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T) \|z_1\|_{C([0, T]; W_0^{1, q}(\Omega))} \leq C(T) \|g\|_{L^q(0, T; L^q(\Omega))}. \quad (2.3)$$

On the other hand, since Δ is the infinitesimal generator of a C_0 semigroup on $C_0(\Omega)$ (See e.g. Theorem 3.7 of Chapter 7 in [12]), we have that

$$\|z_2\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T) \|z_0\|_{C(\overline{\Omega})}, \quad (2.4)$$

which, combined with (2.3) and (2.2), completes the proof. \square

Based on Lemma 2.1, we have the following result.

Proposition 2.2. *Let $a \in L^\infty(\Omega \times (0, T))$, $g \in L^q(0, T; L^q(\Omega))$ and $z_0 \in C_0(\Omega)$. Then the equation*

$$\begin{cases} z_t - \Delta z + az = g & \text{in } \Omega \times (0, T), \\ z = 0 & \text{on } \partial\Omega \times (0, T), \\ z(0) = z_0 & \text{in } \Omega \end{cases} \quad (2.5)$$

has a unique solution, denoted by z , in

$$L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega)) \cap C([0, T]; C_0(\Omega)).$$

Moreover,

$$\|z\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T, \|a\|_{L^\infty(\Omega \times (0, T))}) (\|g\|_{L^q(0, T; L^q(\Omega))} + \|z_0\|_{C(\overline{\Omega})}).$$

Proof. It is obvious that (2.5) has a unique solution, denoted by z , in $L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))$. Moreover,

$$z = z_1 + z_2, \quad (2.6)$$

where z_1 and z_2 are solutions to equations

$$\begin{cases} (z_1)_t - \Delta z_1 + az_1 = g & \text{in } \Omega \times (0, T), \\ z_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ z_1(0) = 0 & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} (z_2)_t - \Delta z_2 + az_2 = 0 & \text{in } \Omega \times (0, T), \\ z_2 = 0 & \text{on } \partial\Omega \times (0, T), \\ z_2(0) = z_0 & \text{in } \Omega, \end{cases} \quad (2.7)$$

respectively.

On one hand, it follows from Sobolev embedding theorem and L^p -theory for parabolic equation (See e.g. Theorem 9.1 of Chapter 4 in [9]) that

$$\|z_1\|_{C([0,T];C(\overline{\Omega}))} \leq C(T)\|z_1\|_{C([0,T];W_0^{1,q}(\Omega))} \leq C(T, \|a\|_{L^\infty(\Omega \times (0,T))})\|g\|_{L^q(0,T;L^q(\Omega))}. \quad (2.8)$$

On the other hand, we set

$$w(t) \triangleq e^{-\|a\|_{L^\infty(\Omega \times (0,T))}t} z_2(t), \quad \forall t \in [0, T]. \quad (2.9)$$

One can easily check that

$$\begin{cases} w_t - \Delta w + (a + \|a\|_{L^\infty(\Omega \times (0,T))})w = 0 & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \partial\Omega \times (0, T), \\ w(0) = z_0 & \text{in } \Omega. \end{cases} \quad (2.10)$$

Let $\varphi \in C([0, T]; C_0(\Omega))$ be the unique solution to

$$\begin{cases} \varphi_t - \Delta \varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(0) = |z_0| & \text{in } \Omega. \end{cases} \quad (2.11)$$

Then

$$\varphi(x, t) \geq 0 \text{ for each } (x, t) \in \Omega \times (0, T), \quad (2.12)$$

and by the same argument for (2.4), we have that

$$\|\varphi\|_{C([0,T];C(\overline{\Omega}))} \leq C(T)\|z_0\|_{C(\overline{\Omega})}. \quad (2.13)$$

By (2.10) and (2.11), we get that

$$\begin{cases} (w - \varphi)_t - \Delta(w - \varphi) + (a + \|a\|_{L^\infty(\Omega \times (0,T))})w = 0 & \text{in } \Omega \times (0, T), \\ w - \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ (w - \varphi)(0) \leq 0 & \text{in } \Omega. \end{cases} \quad (2.14)$$

Multiplying the first equation of (2.14) by $(w - \varphi)^+$ and integrating it over Ω , after some simple calculations, we obtain that

$$\frac{1}{2} \frac{d}{dt} \|(w - \varphi)^+\|_{L^2(\Omega)}^2 + \int_{\Omega} (a + \|a\|_{L^\infty(\Omega \times (0, T))}) w (w - \varphi)^+ dx \leq 0, \quad \text{a.e. } t \in (0, T).$$

Integrating the latter inequality over $(0, t)$ and by (2.12), we have that

$$\|(w - \varphi)^+(t)\|_{L^2(\Omega)} \leq 0 \quad \text{for } t \in (0, T),$$

which implies

$$w(x, t) \leq \varphi(x, t) \quad \text{for a.e. } (x, t) \in \Omega \times (0, T).$$

Similarly, $w(x, t) \geq -\varphi(x, t)$ for a.e. $(x, t) \in \Omega \times (0, T)$. Hence

$$|w(x, t)| \leq \varphi(x, t) \quad \text{for a.e. } (x, t) \in \Omega \times (0, T). \quad (2.15)$$

This together with (2.9) and (2.13) indicates

$$\|z_2\|_{L^\infty(\Omega \times (0, T))} \leq C(T, \|a\|_{L^\infty(\Omega \times (0, T))}) \|z_0\|_{C(\overline{\Omega})},$$

which, combined with (2.7) and Lemma 2.1, implies

$$\|z_2\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T, \|a\|_{L^\infty(\Omega \times (0, T))}) \|z_0\|_{C(\overline{\Omega})}.$$

By the latter inequality, (2.8) and (2.6), we complete the proof. \square

The next result is about the existence and energy estimates of solutions for semilinear heat equations.

Proposition 2.3. *Let $g \in L^q(0, T; L^q(\Omega))$ and $z_0 \in C_0(\Omega)$. The equation*

$$\begin{cases} z_t - \Delta z + f(z) = g & \text{in } \Omega \times (0, T), \\ z = 0 & \text{on } \partial\Omega \times (0, T), \\ z(0) = z_0 & \text{in } \Omega \end{cases} \quad (2.16)$$

has a unique solution, denoted by z , in $C([0, T]; C_0(\Omega))$. Moreover,

$$\|z\|_{C([0, T]; C(\overline{\Omega}))} \leq C(T) (\|g\|_{L^q(0, T; L^q(\Omega))} + \|z_0\|_{C(\overline{\Omega})}). \quad (2.17)$$

Proof. The proof of this proposition is divided into three steps.

Step 1. To show the uniqueness

Suppose that $z_1 \in C([0, T]; C_0(\Omega))$ and $z_2 \in C([0, T]; C_0(\Omega))$ are two solutions of (2.16) on $[0, T]$. Then

$$z_1(t) - z_2(t) = \int_0^t S(t-s) (f(z_2(s)) - f(z_1(s))) ds, \quad \forall t \in [0, T]. \quad (2.18)$$

Here and throughout the proof of this proposition, $S(\cdot)$ is the C_0 semigroup generated by Dirichlet-Laplacian on $L^2(\Omega)$. Denote

$$K \triangleq \max\{\|z_1\|_{C([0,T];C(\overline{\Omega}))}, \|z_2\|_{C([0,T];C(\overline{\Omega}))}\} + 1,$$

and let $L > 0$ be the Lipschitz constant of f on $[-K, K]$. Then it follows from (2.18) that

$$\begin{aligned} \|(z_1 - z_2)(t)\|_{C(\overline{\Omega})} &= \left\| \int_0^t S(t-s)(f(z_1(s)) - f(z_2(s))) ds \right\|_{C(\overline{\Omega})} \\ &\leq \int_0^t \|f(z_1(s)) - f(z_2(s))\|_{C(\overline{\Omega})} ds \\ &\leq L \int_0^t \|z_1(s) - z_2(s)\|_{C(\overline{\Omega})} ds, \quad \forall t \in [0, T], \end{aligned}$$

which, combined with Gronwall's inequality, indicates the uniqueness.

Step 2. To prove the existence

By Lemma 2.1, we have that

$$\int_0^t S(t-s)g(s) ds \in C([0, T]; C_0(\Omega)).$$

Let $\tilde{K} \triangleq \|z_0\|_{C(\overline{\Omega})} + 1 + \left\| \int_0^t S(t-s)g(s) ds \right\|_{C([0,T];C(\overline{\Omega}))}$ and let $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ be defined by:

$$\tilde{f}(r) \triangleq \begin{cases} f(\tilde{K}) & \text{if } r > \tilde{K}, \\ f(r) & \text{if } |r| \leq \tilde{K}, \\ f(-\tilde{K}) & \text{if } r < -\tilde{K}. \end{cases} \quad (2.19)$$

Then the function \tilde{f} is globally Lipschitz. By Lemma 2.1, the following equation

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} + \tilde{f}(\tilde{w}) = g & \text{in } \Omega \times (0, T), \\ \tilde{w} = 0 & \text{on } \partial\Omega \times (0, T), \\ \tilde{w}(0) = z_0 & \text{in } \Omega \end{cases} \quad (2.20)$$

has a unique solution $\tilde{w} \in C([0, T]; C_0(\Omega))$. Moreover,

$$\tilde{w}(t) = S(t)z_0 + \int_0^t S(t-s)(g(s) - \tilde{f}(\tilde{w}(s))) ds, \quad \forall t \in [0, T].$$

Then

$$\begin{aligned} &\|\tilde{w}(t)\|_{C(\overline{\Omega})} \\ &\leq \|S(t)z_0\|_{C(\overline{\Omega})} + \left\| \int_0^t S(t-s)g(s) ds \right\|_{C([0,T];C(\overline{\Omega}))} + \int_0^t \|\tilde{f}(\tilde{w}(s))\|_{C(\overline{\Omega})} ds \\ &\leq \|z_0\|_{C(\overline{\Omega})} + \left\| \int_0^t S(t-s)g(s) ds \right\|_{C([0,T];C(\overline{\Omega}))} + t\|f\|_{C([- \tilde{K}, \tilde{K}])}, \quad \forall t \in [0, T]. \end{aligned}$$

Set $T_1 = \min \left\{ T, \|f\|_{C([-K, K])}^{-1} \right\}$. From the latter it follows that

$$\|\tilde{w}\|_{C([0, T_1]; C(\overline{\Omega}))} \leq \tilde{K},$$

which, combined with (2.19) and (2.20), indicates the equation

$$\begin{cases} \tilde{z}_t - \Delta \tilde{z} + f(\tilde{z}) = g & \text{in } \Omega \times (0, T), \\ \tilde{z} = 0 & \text{on } \partial\Omega \times (0, T), \\ \tilde{z}(0) = z_0 & \text{in } \Omega. \end{cases} \quad (2.21)$$

has a solution in $C([0, T_1]; C_0(\Omega))$. Then by iterating this construction and the uniqueness as Step 1, we obtain the existence of a solution \tilde{z} on $[0, T]$ or on a maximal time interval $[0, T_{\max})$ with $T_{\max} \leq T$.

Case 1. $\tilde{z} \in C([0, T]; C_0(\Omega))$. In this case, the proof is finished.

Case 2. $\tilde{z} \in C([0, T_{\max}); C_0(\Omega))$. We claim that this case cannot happen.

By contradiction, on one hand, since $[0, T_{\max})$ is the maximal time interval for the solution of (2.21) in $C([0, T_{\max}); C_0(\Omega))$, we have

$$\limsup_{t \uparrow T_{\max}} \|\tilde{z}(t)\|_{C(\overline{\Omega})} = +\infty. \quad (2.22)$$

On the other hand, for any $\tilde{T} \in (0, T_{\max})$, we consider the following two equations:

$$\begin{cases} \tilde{z}_t - \Delta \tilde{z} + f(\tilde{z}) = g & \text{in } \Omega \times (0, \tilde{T}), \\ \tilde{z} = 0 & \text{on } \partial\Omega \times (0, \tilde{T}), \\ \tilde{z}(0) = z_0 & \text{in } \Omega \end{cases} \quad (2.23)$$

and

$$\begin{cases} \psi_t - \Delta \psi = |g| & \text{in } \Omega \times (0, T), \\ \psi = 0 & \text{on } \partial\Omega \times (0, T), \\ \psi(0) = |z_0| & \text{in } \Omega. \end{cases} \quad (2.24)$$

It follows from (2.23) and (2.24) that

$$\begin{cases} (\tilde{z} - \psi)_t - \Delta(\tilde{z} - \psi) + f(\tilde{z}) \leq 0 & \text{in } \Omega \times (0, \tilde{T}), \\ \tilde{z} - \psi = 0 & \text{on } \partial\Omega \times (0, \tilde{T}), \\ (\tilde{z} - \psi)(0) \leq 0 & \text{in } \Omega. \end{cases} \quad (2.25)$$

Multiplying the first equation of (2.25) by $(\tilde{z} - \psi)^+$ and integrating it over Ω , by (H_2) and the similar arguments for (2.15), we have that

$$|\tilde{z}(x, t)| \leq \psi(x, t) \text{ a.e. in } (x, t) \in \Omega \times (0, \tilde{T}), \quad (2.26)$$

which, combined with (2.24) and Lemma 2.1, indicates

$$\begin{aligned} \|\tilde{z}\|_{C([0,\tilde{T}];C(\overline{\Omega}))} &\leq \|\psi\|_{C([0,\tilde{T}];C(\overline{\Omega}))} \leq \|\psi\|_{C([0,T];C(\overline{\Omega}))} \\ &\leq C(T)(\|g\|_{L^q(0,T;L^q(\Omega))} + \|z_0\|_{C(\overline{\Omega})}). \end{aligned} \quad (2.27)$$

This contradicts (2.22).

Step 3. To show (2.17)

Consider the following equation

$$\begin{cases} \phi_t - \Delta\phi = |g| & \text{in } \Omega \times (0, T), \\ \phi = 0 & \text{on } \partial\Omega \times (0, T), \\ \phi(0) = |z_0| & \text{in } \Omega. \end{cases}$$

By the similar arguments led to (2.27), we have

$$\|z\|_{C([0,T];C(\overline{\Omega}))} \leq \|\phi\|_{C([0,T];C(\overline{\Omega}))} \leq C(T)(\|g\|_{L^q(0,T;L^q(\Omega))} + \|z_0\|_{C(\overline{\Omega})}).$$

This completes the proof. \square

3 Proof of Theorem 1.1

In this section, we shall first present the following local null controllability of a semilinear heat equation.

Proposition 3.1. *Let $E \subset (0, T)$ with $|E| > 0$. For any $\phi \in L^\infty(\Omega \times (0, T))$, there are two positive constants $\rho_0 \triangleq \rho_0(E, T, f, \|\phi\|_{L^\infty(\Omega \times (0, T))})$ and $\kappa \triangleq \kappa(E, T, f, \|\phi\|_{L^\infty(\Omega \times (0, T))})$, so that for any $w_0 \in C_0(\Omega)$ with $\|w_0\|_{C(\overline{\Omega})} \leq \rho_0$, there exists a function $v \in L^\infty(\Omega \times (0, T))$ so that*

$$\|v\|_{L^\infty(\Omega \times (0, T))} \leq \kappa \|w_0\|_{L^2(\Omega)},$$

and $w \in C([0, T]; C_0(\Omega))$ satisfies

$$\begin{cases} w_t - \Delta w + f(\phi + w) - f(\phi) = \chi_\omega \chi_E v & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \partial\Omega \times (0, T), \\ w(0) = w_0 & \text{in } \Omega, \\ w(T) = 0 & \text{in } \Omega. \end{cases}$$

Proof. By a classical density argument, we may assume that $f \in C^1$. We shall use the Kakutani's Fixed Point Theorem to prove it. To this end, for any $(x, t) \in \Omega \times (0, T)$, we define

$$a(x, t, r) \triangleq \begin{cases} \frac{f(\phi(x, t) + r) - f(\phi(x, t))}{r} & \text{if } r \neq 0, \\ f'(\phi(x, t)) & \text{if } r = 0. \end{cases}$$

Set

$$\mathcal{K} \triangleq \{\xi \in L^2(0, T; L^2(\Omega)) : \|\xi\|_{L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))} + \|\xi\|_{L^\infty(\Omega \times (0, T))} \leq 1\}.$$

For each $\xi \in \mathcal{K}$, consider the linear control system

$$\begin{cases} w_t - \Delta w + a(x, t, \xi(x, t))w = \chi_\omega \chi_E v & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \partial\Omega \times (0, T), \\ w(0) = w_0 & \text{in } \Omega. \end{cases} \quad (3.1)$$

By (H_1) , we have that

$$|a(x, t, \xi(x, t))| \leq C_1 \text{ a.e. } (x, t) \in \Omega \times (0, T), \quad (3.2)$$

where

$$C_1 \triangleq C_1(f, \|\phi\|_{L^\infty(\Omega \times (0, T))})$$

denotes the Lipschitz constant of f on $[-\|\phi\|_{L^\infty(\Omega \times (0, T))} - 1, \|\phi\|_{L^\infty(\Omega \times (0, T))} + 1]$. By Lemma 1.2, (3.1) and (3.2), there exists

$$\kappa \triangleq \kappa(E, T, f, \|\phi\|_{L^\infty(\Omega \times (0, T))}) > 0 \text{ and } v \in L^\infty(\Omega \times (0, T))$$

with

$$\|v\|_{L^\infty(\Omega \times (0, T))} \leq \kappa \|w_0\|_{L^2(\Omega)}, \quad (3.3)$$

so that

$$w(T) = 0. \quad (3.4)$$

Now, we define the multivalued map $\Phi : \mathcal{K} \rightarrow L^2(0, T; L^2(\Omega))$ by

$$\Phi(\xi) \triangleq \{w : \text{there exists a control } v \text{ so that (3.1), (3.3) and (3.4) hold}\}, \text{ for } \xi \in \mathcal{K}.$$

From the above arguments it follows that $\Phi(\xi) \neq \emptyset$ for each $\xi \in \mathcal{K}$.

Next we check in three steps the conditions of Kakutani's fixed point theorem.

Step 1. To show that \mathcal{K} is a convex and compact set in $L^2(0, T; L^2(\Omega))$ and $\Phi(\xi)$ is a convex set in $L^2(0, T; L^2(\Omega))$ for each $\xi \in \mathcal{K}$

These can be directly checked.

Step 2. To show that $\Phi(\mathcal{K}) \subset \mathcal{K}$

To achieve this goal, we observe that for every $\xi \in \mathcal{K}$, there is $v \in L^\infty(\Omega \times (0, T))$, with the estimate

$$\|v\|_{L^\infty(\Omega \times (0, T))} \leq \kappa \|w_0\|_{L^2(\Omega)}, \quad (3.5)$$

so that the associated state $w = w(x, t)$ satisfies

$$\begin{cases} w_t - \Delta w + a(x, t, \xi(x, t))w = \chi_\omega \chi_E v & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \partial\Omega \times (0, T), \\ w(0) = w_0 & \text{in } \Omega, \\ w(T) = 0 & \text{in } \Omega. \end{cases} \quad (3.6)$$

The standard energy method, the fact that $|a| \leq C_1$ and Proposition 2.2, as well as (3.5) and (3.6), lead to

$$\|w\|_{L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))} + \|w\|_{C([0, T]; C(\overline{\Omega}))} \leq C_2 \|w_0\|_{C(\overline{\Omega})},$$

for some positive constant $C_2 \triangleq C_2(E, T, f, \|\phi\|_{L^\infty(\Omega \times (0, T))})$. Hence, if

$$\|w_0\|_{C(\overline{\Omega})} \leq \rho_0 \triangleq C_2^{-1},$$

then $\Phi(\mathcal{K}) \subset \mathcal{K}$.

Step 3. To show that Φ is upper semicontinuous in $L^2(0, T; L^2(\Omega))$

It suffices to show that if

$$\xi_m \in \mathcal{K} \rightarrow \xi \text{ strongly in } L^2(0, T; L^2(\Omega))$$

and

$$w_m \in \Phi(\xi_m) \rightarrow w \text{ strongly in } L^2(0, T; L^2(\Omega)),$$

then $w \in \Phi(\xi)$. To this end, we first observe that $\xi \in \mathcal{K}$. Next we claim that there exists a subsequence of $\{\xi_m\}_{m \geq 1}$, still denoted in the same manner, so that

$$a(x, t, \xi_m)w_m \rightarrow a(x, t, \xi)w \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (3.7)$$

Indeed, since $\xi_m \rightarrow \xi$ strongly in $L^2(0, T; L^2(\Omega))$, we have a subsequence of $\{\xi_m\}_{m \geq 1}$, still denoted by itself, so that

$$\xi_m(x, t) \rightarrow \xi(x, t) \text{ for a.e. } (x, t) \in \Omega \times (0, T).$$

On one hand, for (x, t) with $\xi(x, t) \neq 0$, by the above, there exists a positive integer m_0 depending on (x, t) so that

$$\xi_m(x, t) \neq 0 \quad \forall m \geq m_0,$$

which, combined with the definition of a , implies that

$$a(x, t, \xi_m(x, t)) \rightarrow a(x, t, \xi(x, t)) \text{ as } m \rightarrow +\infty. \quad (3.8)$$

On the other hand, for any (x, t) satisfying $\xi(x, t) = 0$, by the definition of a , we have that $a(x, t, \xi(x, t)) = f'(\phi(x, t))$. Since

$$a(x, t, \xi_m(x, t)) = \begin{cases} \frac{f(\phi(x, t) + \xi_m(x, t)) - f(\phi(x, t))}{\xi_m(x, t)} & \text{if } \xi_m(x, t) \neq 0, \\ f'(\phi(x, t)) & \text{if } \xi_m(x, t) = 0, \end{cases}$$

it holds that

$$a(x, t, \xi_m(x, t)) \rightarrow a(x, t, \xi(x, t)) \text{ as } m \rightarrow +\infty.$$

This, combined with (3.8), implies

$$a(x, t, \xi_m(x, t)) \rightarrow a(x, t, \xi(x, t)) \text{ for a.e. } (x, t) \in \Omega \times (0, T).$$

From the latter, the fact that $|a(x, t, \xi_m(x, t))| \leq C_1$ and Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} & \|a(x, t, \xi_m)w_m - a(x, t, \xi)w\|_{L^2(0, T; L^2(\Omega))}^2 \\ & \leq 2\|a(x, t, \xi_m)(w_m - w)\|_{L^2(0, T; L^2(\Omega))}^2 + 2\|(a(x, t, \xi_m) - a(x, t, \xi))w\|_{L^2(0, T; L^2(\Omega))}^2 \\ & \leq 2C_1^2\|w_m - w\|_{L^2(0, T; L^2(\Omega))}^2 + 2\|(a(x, t, \xi_m) - a(x, t, \xi))w\|_{L^2(0, T; L^2(\Omega))}^2 \\ & \rightarrow 0. \end{aligned}$$

This leads to (3.7).

Finally, since $w_m \in \Phi(\xi_m) \subset \mathcal{K}$, there are $v_m \in L^\infty(\Omega \times (0, T))$, $m = 1, 2, \dots$, satisfying

$$\|v_m\|_{L^\infty(\Omega \times (0, T))} \leq \kappa\|w_0\|_{L^2(\Omega)} \text{ for all } m \geq 1, \quad (3.9)$$

$$\begin{cases} (w_m)_t - \Delta w_m + a(x, t, \xi_m(x, t))w_m = \chi_\omega \chi_E v_m & \text{in } \Omega \times (0, T), \\ w_m = 0 & \text{on } \partial\Omega \times (0, T), \\ w_m(0) = w_0 & \text{in } \Omega, \\ w_m(T) = 0 & \text{in } \Omega \end{cases} \quad (3.10)$$

and

$$\|w_m\|_{L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))} + \|w_m\|_{L^\infty(\Omega \times (0, T))} \leq 1. \quad (3.11)$$

Thus, there is a control v and a subsequence of $\{m\}_{m \geq 1}$, still denoted by itself, so that

$$v_m \rightarrow v \text{ weakly star in } L^\infty(\Omega \times (0, T)), \quad (3.12)$$

$$\begin{aligned} w_m & \rightarrow w \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega)), \\ & \text{weakly star in } L^\infty(\Omega \times (0, T)), \end{aligned} \quad (3.13)$$

and

$$w_m(T) \rightarrow w(T) \text{ strongly in } L^2(\Omega). \quad (3.14)$$

Finally, passing to the limit for $m \rightarrow +\infty$ in (3.9)-(3.11), making use of (3.7) and (3.12)-(3.14), we obtain that $w \in \Phi(\xi)$.

Now, by conclusions in Step 1-Step 3, we can apply Kakutani's fixed point theorem to get a function $w \in \mathcal{K}$ so that $w \in \Phi(w)$, which, combined with Proposition 2.2, indicates $w \in C([0, T]; C_0(\Omega))$. Moreover, since

$$a(x, t, w(x, t))w(x, t) = f(\phi(x, t) + w(x, t)) - f(\phi(x, t)),$$

the results follow at once. This completes the proof of this proposition. \square

Remark 3.2. *In this proposition, only the hypothesis (H_1) is used.*

The next proposition is concerned with the existence of admissible controls for the problem (P) .

Proposition 3.3. *There exists an admissible control for the problem (P) .*

Proof. For any $T_0 > 0$ fixed, we consider the following equation

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } \Omega \times (0, T_0 + 1), \\ y = 0 & \text{on } \partial\Omega \times (0, T_0 + 1), \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \quad (3.15)$$

It follows from Proposition 2.3 that (3.15) has a unique solution in $C([0, T_0 + 1]; C_0(\Omega)) \cap L^2(0, T_0 + 1; H_0^1(\Omega)) \cap W^{1,2}(0, T_0 + 1; H^{-1}(\Omega))$, denoted it by $y(\cdot; y_0, 0)$. Multiplying the first equation of (3.15) by $y(t; y_0, 0)$ and integrating it over Ω , by (H_2) , we get that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |y(t; y_0, 0)|^2 dx + \int_{\Omega} |\nabla y(t; y_0, 0)|^2 dx \leq 0,$$

which implies

$$\frac{d}{dt} \|y(t; y_0, 0)\|_{L^2(\Omega)}^2 + 2\lambda_1 \|y(t; y_0, 0)\|_{L^2(\Omega)}^2 \leq 0 \quad \text{for a.e. } t \in (0, T_0 + 1).$$

Here $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with zero boundary condition. Multiplying the above inequality by $e^{2\lambda_1 t}$ and integrating it over $(0, t)$, after some calculations, we obtain that

$$\|y(T_0; y_0, 0)\|_{L^2(\Omega)} \leq e^{-\lambda_1 T_0} \|y_0\|_{L^2(\Omega)} \quad (3.16)$$

and

$$\|y(T_0 + 1; y_0, 0)\|_{L^2(\Omega)} \leq e^{-\lambda_1(T_0+1)} \|y_0\|_{L^2(\Omega)}. \quad (3.17)$$

Set

$$w(t) \triangleq y(t + T_0; y_0, 0), \quad \forall t \in [0, 1]. \quad (3.18)$$

Then $w \in C([0, 1]; C_0(\Omega))$ satisfies

$$\begin{cases} w_t - \Delta w + f(w) = 0 & \text{in } \Omega \times (0, 1), \\ w = 0 & \text{on } \partial\Omega \times (0, 1), \\ w(0) = y(T_0; y_0, 0) & \text{in } \Omega. \end{cases}$$

By the similar arguments used to obtain (2.26), we have

$$\|w(1)\|_{C(\overline{\Omega})} \leq \|S(1)y(T_0; y_0, 0)\|_{C(\overline{\Omega})}. \quad (3.19)$$

Here and throughout this Proposition, $S(\cdot)$ is the C_0 semigroup generated by Dirichlet-Laplacian on $L^2(\Omega)$. It follows from Proposition 4.4 of Chapter 1 in [1] that

$$\|S(1)y(T_0; y_0, 0)\|_{C(\overline{\Omega})} \leq C \|y(T_0; y_0, 0)\|_{L^1(\Omega)}.$$

Here $C > 0$ is a constant independent of T_0 . From the latter inequality, (3.18), (3.19) and (3.16), we obtain that

$$\|y(T_0 + 1; y_0, 0)\|_{C(\overline{\Omega})} \leq C|\Omega|^{\frac{1}{2}} e^{-\lambda_1 T_0} \|y_0\|_{L^2(\Omega)}. \quad (3.20)$$

By Proposition 3.1, there exist two positive constants $\rho_0 \triangleq \rho_0(f)$ and $\kappa \triangleq \kappa(f)$, so that for any $\varphi_0 \in C_0(\Omega)$ with $\|\varphi_0\|_{C(\overline{\Omega})} \leq \rho_0$, there exists a function $v \in L^\infty(0, 1; L^q(\Omega))$ with

$$\|v\|_{L^\infty(0,1;L^q(\Omega))} \leq \kappa \|\varphi_0\|_{L^2(\Omega)}, \quad (3.21)$$

so that $\varphi \in C([0, 1]; C_0(\Omega))$ satisfies

$$\begin{cases} \varphi_t - \Delta \varphi + f(\varphi) = \chi_\omega v & \text{in } \Omega \times (0, 1), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, 1), \\ \varphi(0) = \varphi_0 & \text{in } \Omega, \\ \varphi(1) = 0 & \text{in } \Omega. \end{cases} \quad (3.22)$$

Now, we take

$$T_0 \triangleq \frac{1}{\lambda_1} \left(\left| \ln \frac{C|\Omega|^{\frac{1}{2}} \|y_0\|_{L^2(\Omega)}}{\rho_0} \right| + \left| \ln \frac{\kappa \|y_0\|_{L^2(\Omega)}}{M} \right| \right) + 1.$$

Then it follows from (3.17) and (3.20)-(3.22) that there exists a control $u \in L^\infty(0, 1; L^q(\Omega))$ with

$$\|u\|_{L^\infty(0,1;L^q(\Omega))} \leq M,$$

so that $z \in C([0, 1]; C_0(\Omega))$ satisfies

$$\begin{cases} z_t - \Delta z + f(z) = \chi_\omega u & \text{in } \Omega \times (0, 1), \\ z = 0 & \text{on } \partial\Omega \times (0, 1), \\ z(0) = y(T_0 + 1; y_0, 0) & \text{in } \Omega, \\ z(1) = 0 & \text{in } \Omega. \end{cases}$$

Finally, it is easy to check the function defined by

$$\overline{u}(t) \triangleq \begin{cases} 0, & t \in (0, T_0 + 1), \\ u(t - T_0 - 1), & t \in (T_0 + 1, T_0 + 2), \\ 0, & t \in (T_0 + 2, +\infty), \end{cases}$$

is an admissible control for the problem (P). \square

Based on Proposition 3.3, we shall show the existence of optimal solutions for the problem (P).

Proposition 3.4. *There exists at least one optimal solution for the problem (P).*

Proof. Let $T^* \triangleq \inf(P)$. It is obvious that $0 \leq T^* < +\infty$. Then there exist sequences $\{T_n\}_{n \geq 1}$ and $\{u_n\}_{n \geq 1} \subset \mathcal{U}$ so that

$$T^* = \lim_{n \rightarrow +\infty} T_n \quad (3.23)$$

and

$$\begin{cases} \partial_t y_n - \Delta y_n + f(y_n) = \chi_\omega u_n & \text{in } \Omega \times (0, T_n), \\ y_n = 0 & \text{on } \partial\Omega \times (0, T_n), \\ y_n(0) = y_0 & \text{in } \Omega, \\ y_n(T_n) = 0 & \text{in } \Omega, \end{cases} \quad (3.24)$$

where $y_n(\cdot) \in C([0, T_n]; C_0(\Omega))$. By (3.23) and (3.24), we can assume that $0 < T_n < T^* + 1$. Set

$$v_n(t) \triangleq \begin{cases} u_n(t), & t \in [0, T_n), \\ 0, & t \in [T_n, +\infty) \end{cases} \quad \text{and} \quad z_n(t) \triangleq \begin{cases} y_n(t), & t \in [0, T_n), \\ 0, & t \in [T_n, T^* + 1]. \end{cases} \quad (3.25)$$

Since $\{u_n\}_{n \geq 1} \subset \mathcal{U}$, it follows from (3.24) and (3.25) that

$$\{v_n\}_{n \geq 1} \subset \mathcal{U} \quad (3.26)$$

and that $z_n(\cdot) \in C([0, T^* + 1]; C_0(\Omega))$ satisfies

$$\begin{cases} \partial_t z_n - \Delta z_n + f(z_n) = \chi_\omega v_n & \text{in } \Omega \times (0, T^* + 1), \\ z_n = 0 & \text{on } \partial\Omega \times (0, T^* + 1), \\ z_n(0) = y_0 & \text{in } \Omega, \\ z_n(T_n) = 0 & \text{in } \Omega. \end{cases} \quad (3.27)$$

By (3.27), (3.26) and Proposition 2.3, we obtain that

$$\begin{aligned} & \|z_n\|_{L^2(0, T^* + 1; H_0^1(\Omega)) \cap W^{1,2}(0, T^* + 1; H^{-1}(\Omega))} + \|z_n\|_{C([0, T^* + 1]; C(\overline{\Omega}))} \\ & \leq C(\|v_n\|_{L^q(0, T^* + 1; L^q(\Omega))} + \|y_0\|_{C(\overline{\Omega})}) \\ & \leq C, \quad \forall n \geq 1. \end{aligned}$$

Here $C > 0$ is a constant independent of n . This, combined with (3.26) and (3.27), implies that there exists a subsequence of $\{n\}_{n \geq 1}$, denoted in the same manner, $z \in L^2(0, T^* + 1; H_0^1(\Omega)) \cap W^{1,2}(0, T^* + 1; H^{-1}(\Omega)) \cap L^\infty(\Omega \times (0, T^* + 1))$ and $v \in \mathcal{U}$, so that

$$\begin{aligned} z_n & \rightarrow z \quad \text{weakly in } L^2(0, T^* + 1; H_0^1(\Omega)) \cap W^{1,2}(0, T^* + 1; H^{-1}(\Omega)), \\ & \quad \text{weakly star in } L^\infty(\Omega \times (0, T^* + 1)) \text{ and strongly in } C([0, T^* + 1]; L^2(\Omega)), \\ v_n & \rightarrow v \quad \text{weakly star in } L^\infty(0, +\infty; L^q(\Omega)). \end{aligned} \quad (3.28)$$

Moreover, $v \in \mathcal{U}$ and z satisfies

$$\begin{cases} \partial_t z - \Delta z + f(z) = \chi_\omega v & \text{in } \Omega \times (0, T^* + 1), \\ z = 0 & \text{on } \partial\Omega \times (0, T^* + 1), \\ z(0) = y_0 & \text{in } \Omega. \end{cases} \quad (3.29)$$

Finally, it follows from Proposition 2.3, the fourth equation of (3.27) and (3.28) that

$$z \in C([0, T^* + 1]; C_0(\Omega))$$

and

$$\begin{aligned} \|z(T^*)\|_{L^2(\Omega)} &\leq \|z(T^*) - z(T_n)\|_{L^2(\Omega)} + \|z(T_n) - z_n(T_n)\|_{L^2(\Omega)} \\ &\leq \|z(T^*) - z(T_n)\|_{L^2(\Omega)} + \|z - z_n\|_{C([0, T^* + 1]; L^2(\Omega))} \rightarrow 0, \end{aligned}$$

which, combined with (3.29), complete the proof. \square

At the end of this section, we give the proof of Theorem 1.1.

Proof. By contradiction, there would exist an optimal control u^* , a positive constant $\varepsilon_0 < M$ and a measurable subset $E^* \subset (0, T^*)$ with $|E^*| > 0$ so that

$$\|u^*(t)\|_{L^q(\Omega)} \leq M - \varepsilon_0, \quad \forall t \in E^*. \quad (3.30)$$

Let $\delta_0 \in (0, |E^*|/2)$ and write

$$E_{\delta_0}^* \triangleq \{t \in (0, T^*) : t + \delta_0 \in E^*\}.$$

One can easily check that

$$|E_{\delta_0}^*| = |E^* \cap (\delta_0, T^*)| \geq |E^*| - \delta_0 > 2^{-1}|E^*| > 0.$$

Denote $y^*(t) \triangleq y(t; y_0, u^*)$ and $z_{\delta_0}^*(t) \triangleq y^*(t + \delta_0)$. Then it holds that

$$\begin{cases} (z_{\delta_0}^*)_t - \Delta z_{\delta_0}^* + f(z_{\delta_0}^*) = \chi_\omega u^*(t + \delta_0) & \text{in } \Omega \times (0, T^* - \delta_0), \\ z_{\delta_0}^* = 0 & \text{on } \partial\Omega \times (0, T^* - \delta_0), \\ z_{\delta_0}^*(0) = y^*(\delta_0) & \text{in } \Omega, \\ z_{\delta_0}^*(T^* - \delta_0) = 0 & \text{in } \Omega. \end{cases} \quad (3.31)$$

By Proposition 3.1, there are two positive constants

$$\rho_0 \triangleq \rho_0(E_{\delta_0}^*, T^*, \delta_0, f, \|z_{\delta_0}^*\|_{L^\infty(\Omega \times (0, T^* - \delta_0))})$$

and

$$\kappa \triangleq \kappa(E_{\delta_0}^*, T^*, \delta_0, f, \|z_{\delta_0}^*\|_{L^\infty(\Omega \times (0, T^* - \delta_0))}),$$

so that for any $w_0 \in C_0(\Omega)$ with $\|w_0\|_{C(\overline{\Omega})} \leq \rho_0$, there exists a function $v \in L^\infty(0, T^* - \delta_0; L^q(\Omega))$ with

$$\|v\|_{L^\infty(0, T^* - \delta_0; L^q(\Omega))} \leq \kappa \|w_0\|_{L^2(\Omega)}, \quad (3.32)$$

so that $w \in C([0, T^* - \delta_0]; C_0(\Omega))$ satisfies

$$\begin{cases} w_t - \Delta w + f(z_{\delta_0}^* + w) - f(z_{\delta_0}^*) = \chi_\omega \chi_{E_{\delta_0}^*} v & \text{in } \Omega \times (0, T^* - \delta_0), \\ w = 0 & \text{on } \partial\Omega \times (0, T^* - \delta_0), \\ w(0) = w_0 & \text{in } \Omega, \\ w(T^* - \delta_0) = 0 & \text{in } \Omega. \end{cases} \quad (3.33)$$

Now, we choose $\delta \in (0, \delta_0)$ so that

$$\|y^*(\delta) - y^*(\delta_0)\|_{C(\overline{\Omega})} + \|y^*(\delta) - y^*(\delta_0)\|_{L^2(\Omega)} \leq \min\{\rho_0, \kappa^{-1}\varepsilon_0\}.$$

This, together with (3.32) and (3.33), indicates that there exists a control $v_\delta \in L^\infty(0, T^* - \delta_0; L^q(\Omega))$ with

$$\|v_\delta\|_{L^\infty(0, T^* - \delta_0; L^q(\Omega))} \leq \kappa \|y^*(\delta) - y^*(\delta_0)\|_{L^2(\Omega)} \leq \varepsilon_0, \quad (3.34)$$

so that $w_\delta \in C([0, T^* - \delta_0]; C_0(\Omega))$ satisfies

$$\begin{cases} (w_\delta)_t - \Delta w_\delta + f(z_{\delta_0}^* + w_\delta) - f(z_{\delta_0}^*) = \chi_\omega \chi_{E_{\delta_0}^*} v_\delta & \text{in } \Omega \times (0, T^* - \delta_0), \\ w_\delta = 0 & \text{on } \partial\Omega \times (0, T^* - \delta_0), \\ w_\delta(0) = y^*(\delta) - y^*(\delta_0) & \text{in } \Omega, \\ w_\delta(T^* - \delta_0) = 0 & \text{in } \Omega. \end{cases} \quad (3.35)$$

It follows from (3.31) and (3.35) that

$$\begin{cases} (z_{\delta_0}^* + w_\delta)_t - \Delta(z_{\delta_0}^* + w_\delta) + f(z_{\delta_0}^* + w_\delta) \\ \quad = \chi_\omega(u^*(t + \delta_0) + \chi_{E_{\delta_0}^*} v_\delta) & \text{in } \Omega \times (0, T^* - \delta_0), \\ z_{\delta_0}^* + w_\delta = 0 & \text{on } \partial\Omega \times (0, T^* - \delta_0), \\ (z_{\delta_0}^* + w_\delta)(0) = y^*(\delta) & \text{in } \Omega, \\ (z_{\delta_0}^* + w_\delta)(T^* - \delta_0) = 0 & \text{in } \Omega. \end{cases} \quad (3.36)$$

Moreover, by (3.30) and (3.34), the function

$$u_\delta^*(t) \triangleq u^*(t + \delta_0) + \chi_{E_{\delta_0}^*} v_\delta(t), \quad t \in (0, T^* - \delta_0), \quad (3.37)$$

satisfies

$$\|u_\delta^*(t)\|_{L^q(\Omega)} \leq M \text{ for a.e. } t \in (0, T^* - \delta_0). \quad (3.38)$$

Finally, we define

$$\tilde{u}_\delta^*(t) \triangleq \begin{cases} u^*(t), & t \in (0, \delta], \\ u_\delta^*(t - \delta), & t \in (\delta, T^* - \delta_0 + \delta], \\ 0, & t \in (T^* - \delta_0 + \delta, +\infty). \end{cases}$$

Then by (3.36)-(3.38) and Proposition 2.3, we obtain that

$$\tilde{u}_\delta^* \in \mathcal{U}, \quad y(\cdot; y_0, \tilde{u}_\delta^*) \in C([0, T^* - \delta_0 + \delta]; C_0(\Omega)) \text{ and } y(T^* - \delta_0 + \delta; y_0, \tilde{u}_\delta^*) = 0.$$

These contradict the optimality of T^* and complete the proof. \square

References

- [1] V.Barbu, Analysis and Control of Nonlinear Infinite Dimensional Systems, Academic Press, Boston, 1993.
- [2] V.Barbu, The time optimal control of Navier-Stokes equations, Systems and Control Letters, 30(1997), 93-100.
- [3] H.O.Fattorini, Time optimal control of solutions of operational differential equations, SIAM Journal on Control, 2(1964), 54-59.
- [4] H.O.Fattorini, Infinite Dimensional Linear Control Systems: The Time Optimal and Norm Optimal Problems, North-Holland Mathematics Studies 201, ELSEVIER, 2005.
- [5] K.Kunisch and L.J.Wang, Time optimal controls of the linear Fitzhugh-Nagumo equation with pointwise control constraints, Journal of Mathematical Analysis and Applications, 395(2012), 114-130.
- [6] K.Kunisch and L.J.Wang, Time optimal control of the heat equation with pointwise control constraints, ESAIM. Control, Optimisation and Calculus of Variations, 19(2013), 460-485.
- [7] K.Kunisch and L.J.Wang, The bang-bang property of time optimal controls for the Burgers equation, Discrete and Continuous Dynamical Systems, 34(2014), Series A, 3611-3637.
- [8] K.Kunisch and L.J.Wang, Bang-bang property of time optimal controls of semilinear parabolic equations, Discrete and Continuous Dynamical Systems, 36(2016), Series A, 279-302.
- [9] O.A.Ladyženskaja, V.A.Solonnikov and N.N.Ural'ceva, Linear and Quasilinear Equations of parabolic Type, American Mathematical Society, 1968.
- [10] J.L.Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1971.
- [11] V.J.Mizel and T.I.Seidman, An abstract bang-bang principle and time optimal boundary control of the heat equation, SIAM Journal on Control and Optimization, 35(1997), 1204-1216.
- [12] A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
- [13] K.D.Phung and G.S.Wang, An observability estimate for parabolic equations from a measurable set in time and its applications, Journal of the European Mathematical Society, 15(2013), 681-703.

- [14] K.D.Phung, L.J.Wang and C.Zhang, Bang-bang property for time optimal control of semi-linear heat equation, *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire*, 31(2014), 477-499.
- [15] E.J.P.G.Schmidt, The “bang-bang” principle for the time-optimal problem in boundary control of the heat equation, *SIAM Journal on Control and Optimization*, 18(1980), 101-107.
- [16] G.S.Wang, L^∞ -null controllability for the heat equation and its consequences for the time optimal control problem, *SIAM Journal on Control and Optimization*, 47(2008), 1701-1720.
- [17] G.S.Wang, Y.S.Xu and Y.B.Zhang, Attainable subspaces and the bang-bang property of time optimal controls for heat equations, *SIAM Journal on Control and Optimization*, 53(2015), 592-621.
- [18] G.S.Wang and L.J.Wang, The Bang-Bang principle of time optimal controls for the heat equation with internal controls, *Systems and Control Letters*, 56(2007), 709-713.
- [19] L.J.Wang and G.S.Wang, The optimal time control of a phase-field system, *SIAM Journal on Control and Optimization*, 42(2003), 1483-1508.